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# Logical Reformulation of Quantum Mechanics. I. Foundations

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The basic rules of quantum mechanics are reformulated. They deal primarily with individual systems and do not assume that every ket may represent a physical state. The customary kinematic and dynamic rules then allow to construct consistent Boolean logics describing the history of a system, following essentially Griffiths' proposal. Logical implication is defined within these logics, the multiplicity of which reflects the complementary principle. Only one interpretative rule of quantum mechanics is necessary in such a framework. It states that these logics provide *bona fide* foundations for the description of a quantum system and for reasoning about it. One attempts to build up classical physics, including classical logic, on these quantum foundations. The resulting theory of measurement needs not to state *a priori* that the eigenvalues of an observable have to be the results of individual measurements nor to assume wave packet reduction. Both these properties can be obtained as consequences of the basic rules. One also needs not to postulate that every observable is measurable, even in principle. A proposition calculus is obtained, allowing in principle the replacement of the discussion of problems concerned with the practical interpretation of experiments by due calculations.

**KEY WORDS:** Quantum mechanics; foundations; logic; classical limit; wave packet reduction.

## 1. INTRODUCTION

Quantum mechanics is a very deep and difficult theory. Despite its tremendous practical success, its conceptual framework raises great difficulties. Some creative minds discovered it and molded it, but some exacting minds, often the same ones, could not entirely accept it.

Anybody who tries to improve or just to modify a minor point in its foundations soon feels even better how tight the theory is and how deeply

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it was thought out. The present work began with such a minor modification. This led to new questions and, surprisingly, to apparently new results and even a new approach, sometimes rather different from the accepted point of view. Things could not begin to clarify until a systematic restatement of the basic rules was attempted. The result is the present paper, in which I do not claim to propose a full-fledged theory, but to offer an apparently consistent approach to criticism.

In Section 2, the kinematic, dynamical, and descriptive rules used to define the Hilbert space, observables, the Schrödinger equation, and physical states are given. Nothing is supposed about classical physics and therefore also nothing about measurement. However, I take seriously the standpoint that classical physics (meaning both classical logic and classical dynamics) should ultimately rest upon quantum foundations. This is why I start from individual systems and not from statistical ensembles and why I do not assume that every ket can represent a physical state.

I start from this general framework, which looks still more abstract than usual from a lack of interpretative rules, and begin a logical construction aiming at a more intuitive understanding of a system by descriptive propositions. The first step, as given in Section 3, is von Neumann's construction for propositions concerning one observable at a given time, together with their associated mathematical measures.[1] I refrain from calling such a measure a probability because experimental trials are not yet available nor even meaningful.

In Section 4, I follow Griffiths' construction to define propositions describing a system at several different times.[2,3] Griffiths' measure for such propositions is recalled in Section 5, with minor modifications. I also recall after him that not all propositions make mathematical sense and I exhibit the consistency relations that select the meaningful families of propositions. These families of propositions are called here consistent quantum representations of (Boolean) logic for reasons to be explained in the text.

In Section 6, I build up some logics and not only descriptive statements by defining a rule of implication among propositions (i.e., the ability to say "if..., then...") by using conditional measures (not yet probabilities). A theorem tells us that we can never meet contradiction although many different representations of logic are at our disposal. This multiplicity of logics may be said to exhibit the many aspects of the complementary principle.

In Section 7, I give a quantum logical meaning to a classical proposition stating that a system is in some large cell of a classical phase space at time $t$. This result is not entirely new. I also state sufficient conditions over the state and the Hamiltonian such that classical dynamics holds and find that, in such a case, classical logic also holds as a limit of quantum logics when Planck's constant may be considered as relatively small. In that section, only some results are given without the proofs. This is because of the somewhat technical character of these proofs, using mathematical microanalysis, which would have made continuous reading uneasy. They are given in a following paper, hereafter denoted by III. Extending these results to macroscopic physical systems raises a host of unsolved problems.

Up to that points no rule has been added to the basic quantum rules of Section 2. I then introduce in Section 8 a very general interpretative rule stating that the consistent quantum representations of logic provide the correct physical way for describing a system and for reasoning about it. I also attempt to show (not being able to prove it entirely) that classical facts may be described by quantum mechanics. Here a fact is defined essentially as some event that leaves a neat physical trace (or a so-called memory), at least for some finite length of time. Admitting these results opens the way to measurement theory.

In Section 9, measurement theory is formulated in the resulting framework. Individual measurements are considered. It is found that only the eigenvalues of an observable can be found in such a measurement and wave packet reduction is also obtained as a consequence. I do not claim that these results could not be obtained otherwise. However, being able now in principle to give a meaning to an individual experimental trial (although I only used the rules of quantum mechanics), it becomes possible to state consistently the last rule interpreting the mathematical measures as probabilities. It should be also mentioned that one does not need to assume that every observable can be measured in principle, either actually or conceptually, although they may enter in propositions.

The case of identical particles is sketched in Section 10.

There is practically no experimental difference between the present formulation of quantum mechanics and the standard ones. Its interest may be, however, twofold: first, to remove or at least to significantly modify some epistemological consequences of standard quantum mechanics that look questionable; second, to provide a proposition calculus that can, in principle, replace any talkative interpretation of an experiment by straightforward calculations.

Some examples of applications are given in the next paper, hereafter called II. Many remain to be done. In the third paper, I give the proofs of the theorems concerning the relation between quantum propositions and classical propositions. They suggest a possible application of the present form of the theory to the problem of irreversibility, a program that is only sketchily described.

Brief accounts of some of these ideas have already been published.[4,5]

## 2. KINEMATIC AND DYNAMICAL RULES

In the present section, I shall give the general rules stating the basic axioms of quantum mechanics as I shall use them, but shall not include the rules allowing an experimental interpretation that will be only obtained later on. I take the notion of an isolated physical system for granted.[6–8]

### 2.1. Kinematics

**Rule 1.** An individual isolated physical system $S$ is described theoretically by a well-defined Hilbert space $\mathscr{H}$ together with the self-adjoint operators acting in $\mathscr{H}$.

*Comments.* The elements of $\mathscr{H}$ will be called vectors and a normed vector will be called a ket. A self-adjoint operator $A$ has a spectrum that is denoted by $\sigma_A$. It is known that $\sigma_A$ generally consists of a discrete part, a continuous measurable part, and a singular part.[1,9] I shall reserve the name *observable* to a self-adjoint operator having no singular spectrum. I shall use Dirac's convention for the spectrum[6]: the eigenvectors of $A$ with eigenvalue $a$ will be denoted by $|a, r\rangle$, where $r$ is a degeneracy index, even when $a$ lies in the continuous spectrum (the necessary rigorous corrections being well known[1,9]). Given a subset $C$ of $\sigma_A$, I shall denote by

$$\int_C da \tag{2.1}$$

a summation over these values, even when the set $C$ is discrete.

I shall treat several commuting observables (as, for instance, the three coordinate operators of a particle position) as only one, considering the spectrum $\sigma$ as having in that case more than one dimension.

### 2.2. States

**Rule 2.** The state of an individual isolated physical system $S$ is specified by a finite-rank positive operator $\rho$ with unit trace that will be called the *state operator.*

*Comments.* Such an operator is, by definition, a self-adjoint operator having a finite number of nonzero positive eigenvalues. The reason for such a rule will become clear when the whole theory has been elaborated, once the preparation of a state will become explicit from measurement theory.

In most cases, it will be convenient to assume that $\rho$ is given in terms of a finite-rank projector $E$ by $\rho = E/(\mathrm{Tr}\, E)$. A special case where the state of $S$ is defined by a ket $|\psi\rangle$ corresponds to $\rho=|\psi\rangle\langle\psi|$. The operator $\rho$ will not be treated as a density operator having only a statistical meaning. Its explicit specification involves the whole past history of $S$, particularly when the system was not yet isolated. When dealing with a nonseparated system, such as the two particles in an Einstein–Podolsky–Rosen situation,[10] I shall include both particles in the isolated system.

I explicitly do not assume that every projector can represent a physical state. In particular, I do not assume that every ket can represent a state. If, for instance, $|b_1\rangle$ could represent the state of a bottle and $|b_2\rangle$ either broken glass or the atoms of the bottle brought into a plasma state, I do not assume that a ket such as $c_1|b_1\rangle+c_2|b_2\rangle$ can represent a physical state.

### 2.3. Dynamics

**Rule 3.** There exists an observable $H$, the Hamiltonian, and an evolution operator $U(t)=\exp(-iHt)$ such that any state operator evolves with time according to $\rho(t)=U(t)\,\rho_0 U^{-1}(t)$, where $\rho_0=\rho(0)$, 0 being a conveniently chosen origin of time.

*N.B.* The rationalized Planck constant $\hbar$ has been taken equal to unity.

### 2.4. Ensembles

One may pass from individual systems to statistical ensembles by defining noninteracting systems. A statistical ensemble is made up of $N$ noninteracting identical copies of a given individual system.

**Rule 4.** Let $S$ and $S'$ be two isolated individual physical systems. The system $\Sigma$ consisting of both $S$ and $S'$ not interacting has for its Hilbert space the tensor product $\mathscr{H}^S\otimes\mathscr{H}^{S'}$. Its hamiltonian is $H\otimes I'+I\otimes H'$. The state operator of $\Sigma$ is the tensor product of the state operators $\rho\otimes\rho'$.

Here $I$ denotes an identity operator.

## 3. ELEMENTARY DESCRIPTION OF A PHYSICAL SYSTEM

The previous rules are very abstract and they need interpretation, i.e., some way to bring them nearer to intuition and to make them experimentally meaningful. The first step will be to let some intuition clarify this purely mathematical framework.

### 3.1. Elementary Predicates

One generally describes a physical situation by some statements such as, "The position of an electron is in a volume $V$ of space." Here the word "electron" may refer to a system $S$ made up of only one electron, the word "position" denotes an observable, and "volume $V$" is a given subset of its spectrum. So, everything is already meaningful except the little word "is." It will be given a place in the theory by associating an observable with the whole sentence.

**Definition 1.** Given a system $S$, an observable $A$, and a subset $C$ of $\sigma_A$, the statement "the observable $A$ is in $C$" will be called an *elementary predicate* and it will be denoted by $[S, A, C]$. When no confusion can arise, it will also be denoted by $[A, C]$ or more simply by $[C]$.

To such a predicate, we associate a projector $F[C]$ (or more simply $F$), which will be called its *predicate projector*, and is defined by

$$F([C]) = \int_C da \sum_r |a, r\rangle\langle a, r| \tag{3.1}$$

I shall denote by $C^*$ the complementary set of $C$ in $\sigma_A$. The predicate $[S, A, C^*]$ will be called the *negation* of the predicate $[S, A, C]$. The associated projectors $F$ and $F^*$ satisfy

$$FF^* = F^*F = 0, \qquad F + F^* = 1 \tag{3.2}$$

### 3.2. The Measure of a Predicate

It will be convenient to introduce here the measure of a set as a mathematical notion. It will be stated without the necessary restriction to Lebesgue-measurable sets, the knowledgeable reader being able to provide the technical notions that play no essential role in the present context.

**Definition 2.** A *measure* $\mu$ on a set $\sigma$ associates a number $\mu(C)$ to any subset $C$ of $\sigma$, satisfying the three following properties[9]:

(i) $\quad \mu(C) \geqslant 0 \qquad (3.3)$

(ii) $\quad \mu(\sigma) = 1 \qquad (3.4)$

(iii) $\quad \mu(C \cup C') = \mu(C) + \mu(C') \quad$ if $\quad C \cap C' = \varnothing \qquad (3.5)$

A measure has all the mathematical properties of a probability. However, it cannot be considered to be a probability in the physical sense as long as one cannot specify what kind of physical experiments (trials) on a statistical ensemble will allow its comparison with the frequency of a result in a series of trials.

**Definition 3.** Given a physical system $S$ with state operator $\rho$ and an elementary predicate $[S, A, C]$, I define the *measure of the predicate in state* $\rho$ by

$$w_\rho(C) = \mathrm{Tr}(\rho F([C])) \tag{3.6}$$

Often I shall omit the index $\rho$ in $w_\rho(C)$. The fact that Eq. (3.6) defines a measure is obvious. When $\rho = |\psi\rangle\langle\psi|$, one has simply

$$w(C) = \int_C da \left( \sum_r |\langle \psi \mid a, r\rangle|^2 \right)$$

I shall now refer the predicates to a given time $t$. To do so, it will be useful to introduce the Heisenberg time-dependent observables that are associated with a given observable $A$ by

$$A(t) = U^{-1}(t)\, A U(t) \tag{3.7}$$

where $A$ may be in particular a projector.

**Definition 4.** Let $S$ be a physical system in a state $\rho_0$ at time zero. The proposition stating that an observable $A$ has its values in a subset $C$ of $\sigma_A$ at time $t$ ($t > 0$) will be called an *elementary predicate at time t* or, more concisely, a predicate and it will be denoted by $[S, A, C, t]$. It is associated with the projector $F(t, C)$, where $F(t, C) = U^{-1}(t)\, F(C)\, U(t)$. To the predicate $[S, A, C, t]$, the system $S$ being in the initial state $\rho_0$, I assign the measure

$$w_{\rho,t}(C) = \mathrm{Tr}(\rho(t)\, F) = \mathrm{Tr}(\rho_0 F(t)) \tag{3.8}$$

where $F = F(C)$.

## 4. HISTORY PREDICATES

It is not enough to be able to describe a physical system at one given time $t$. When we think in particular of a classical object, for instance, we want to consider its whole history when time varies. The first step toward such a goal will be to describe a quantum system at two different times $t_1$ and $t_2$.

### 4.1. History Predicates

A typical example the reader may keep in mind in the next considerations is the following one. Consider an electron. Let $V$ be a volume in configuration space (i.e., a subset in the spectrum of the position operator) and $B$ a similar set in momentum space. I try to assign a meaning to the sentence: "The position of the electron is in $V$ at time $t_1$ and its momentum is in $B$ at time $t_2$." This development is due to Griffiths.[2]

**Definition 5.** Let $S$ be a physical system. Let $A_1$ (resp. $A_2$) be an observable and let $C_1$ be a subset of its spectrum $\sigma_1$ (resp. $C_2$ in $\sigma_2$). The proposition stating that the value of $A_1$ at time $t_1$ is in $C_1$ and the value of $A_2$ at time $t_2$ is in $C_2$ will be called a *history predicate* (sometimes for short a predicate when no confusion can arise). It will be denoted by $[S; A_1, A_2, C_1, C_2, t_1, t_2]$.

The extension to any number of elementary predicates is taken for granted, so that one may speak of the history predicate $[S; A_1,..., A_n, C_1,..., C_n, t_1,..., t_n)]$. I shall always order the elementary predicates with time increasing

$$t_1 < t_2 \cdots < t_n \tag{4.1}$$

The case $t_j = t_{j+1}$ will be considered only when $A_j$ and $A_{j+1}$ commute. In such a case, I shall consider the pair $(A_j, A_{j+1})$ as just one observable, so that the restriction (4.1) will be always assumed.

### 4.2. Families of Propositions

I shall now build up a structure of probabilized space over the history predicates. Let the observables $A_1,..., A_n$ and the times $t_1,..., t_n$ be kept fixed. The direct product $\sigma_1 \times \cdots \times \sigma_n$ of these observables spectra will be denoted by $X$.

Over $X$, I introduce a *basis*. Such a basis is a family $\{D_\alpha\}$ of disjoint subsets covering $X$, i.e.,

$$D_\alpha \cap D_\beta = \varnothing \ (\alpha \neq \beta), \qquad \bigcup_\alpha D_\alpha = X \tag{4.2}$$

Every set $D_\alpha$ will be a "rectangular" set, i.e., a direct product

$$D_\alpha = C_{\alpha 1} \times \cdots \times C_{\alpha n}, \qquad C_{\alpha j} \in \sigma_j \tag{4.3}$$

One can, for instance, fix once and for all a basis of sets $C_{1k},..., C_{pk}$ in each spectrum $\sigma_k$ and take for sets $D_\alpha$ constituting the basis the different direct products $C_{i1} \times \cdots \times C_{ln}$ [in such a case the index $\alpha$ stands for a multi-index

$(i,..., l)$]. In this case, the basis that looks like a brick wall will be said to be of type I or of Griffiths' type. Otherwise (like a brick wall with cornerstones), it will be said to be of type II.

For instance, let $(C_1, C_1^*)$ be a two-set basis of $\sigma_1$ and $(C_2, C_2^*)$ be a basis of $\sigma_2$; then the family of subsets of $X$ given by

$$(C_1 \times C_2), (C_1 \times C_2^*), (C_1^* \times C_2), (C_1^* \times C_2^*) \tag{4.4}$$

is a type I (Griffiths) basis of $X$. On the other hand, the family

$$(C_1 \times C_2), (C_1^* \times C_2), (\sigma_1 \times C_2^*) \tag{4.5}$$

is a type II basis. As indicated by its name, the first type was introduced by Griffiths.[2] I shall use it, but, quite often, it is more convenient to use bases of type II (note that type I is a special case of type II).

A history predicate $[S, A_1,..., A_n, C_{\alpha 1},..., C_{\alpha n}, t_1,..., t_n]$ can be associated with any set $D_\alpha$ in the basis, using (4.3). For brevity it will be denoted by $[S, D_\alpha, t_1,..., t_n]$ or, even more concisely, by $[D_\alpha]$.

Given $X$ and its basis $\{D_\alpha\}$, I shall introduce a Boolean lattice $B$. It includes the empty set and, except for that, it is made up of all possible unions of sets belonging to $\{D_\alpha\}$. $X$ belongs to $B$; union, intersection, and complementation are defined on the sets belonging to $B$ in the ordinary way and these operations give back sets belonging to $B$.

As in probability calculus, one can associate a proposition $[D]$ with any set $D$ belonging to $B$. Conjunction (and, $\wedge$), disjunction (or, $\vee$), and negation (herewith denoted by *) are defined in this family of propositions by

$$[D] \vee [D'] = [D \cup D'], \qquad [D] \wedge [D'] = [D \cap D'], \qquad [D]^* = [D^*] \tag{4.6}$$

When the basis $\{D_\alpha\}$ is of type I, each elementary predicate entering a history predicate belongs automatically to the family of propositions. Take the example of the basis (4.4); then one has, identifying the set $C_1$ in $X$ with $C_1 \times \sigma_2$,

$$C_1 \times \sigma_2 = (C_1 \times C_2) \cup (C_1 \times C_2^*)$$

so that the elementary predicate $[C_1, t_1]$ is a member of the family of propositions that is defined in this respect by

$$[C_1; t_1] = [C_1, C_2; t_1, t_2] \vee [C_1, C_2^*, t_1, t_2]$$

Accordingly, one can also write in that case a history predicate as a logical conjunction of elementary predicates

$$[C_1, C_2, t_1, t_2] = [C_1, t_1] \wedge [C_2, t_2]$$

This is not possible in general for a type II family, where the history predicates cannot always be decomposed into a conjunction of elementary predicates.

Finally, I sum up all these notions:

**Definition 6.** Let $S$ be an individual isolated physical system. Let us be given $n$ observables $A_1,..., A_n$ associated with the times $t_1,..., t_n$ ordered according to (4.1). Let $X = \sigma_1 \times \cdots \times \sigma_n$ be the direct product of their spectra and let $\{D_\alpha\}$ be a basis of $X$ as given by Eq. (4.3). Let $B$ be the Boolean lattice having this basis, so that each set $D$ belonging to $B$ can be written as a union $D_\alpha \cup \cdots \cup D_\lambda$. One associates with $(X, B)$ a family of propositions $[D]$ with conjunction, disjunction, and negation as given by Eq. (4.6). Such a family of proposition will be called a *quantum representation of logic.*

I stress that this is strict Boolean logic and no unconventional logic. What I call here a representation of logic corresponds to what the logicians call an interpretation of abstract logic.[11] The word "interpretation" would be somewhat misleading in the present context, whereas "representation," borrowed from group theory, is used here to emphasize that there is but one abstract logic with many possible realizations and applications. The lattice $B$ is what is called a "universe of discourse" by logicians.[11]

## 5. THE MEASURE OF A HISTORY PROPOSITION

I want now to associate a mathematical measure with a history proposition in order to make a first step toward a later definition of a physical probability. One cannot define it in terms of only one projector as I did in the case of an elementary predicate. However, there is a way out, which was proposed by Griffiths.[2]

### 5.1. Mathematical Measures

First recall the definition of a measure over a Boolean lattice as it is used in mathematics and in probability calculus:

**Definition 7.** Given a set $X$ and a Boolean lattice $B$ on $X$, a *measure* $w$ on $(X, B)$ is a mapping $w: B \to R$ satisfying the three following properties:

(i) $w(D) \geqslant 0$ whatever $D$ in $B$ (5.1)

(ii) $w(X) = 1$ (5.2)

(iii) $w(D \cup D') = w(D) + w(D')$ when $D \cap D' = \varnothing$ (5.3)

I now define such a measure on a lattice of history propositions.

## 5.2. The Measure of a Proposition

To begin with, let us notice some structural aspects of propositions. Any lattice of propositions corresponds to a family of subsets in the space $X = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$. As shown in Fig. 1, the basic subsets or elementary building blocks correspond to an elementary history predicate; a more general proposition corresponds to any union of such building blocks. It appears that some propositions can be history predicates though not elementary, i.e., they are associated with a direct product $D_0 = C_1 \times \cdots \times C_n$, not all the sets $C_k$ being elementary tiles in $\sigma_k$, but the union of several such elementary tiles. Similarly, any proposition corresponds to a unique union of elementary history predicates (i.e., to a

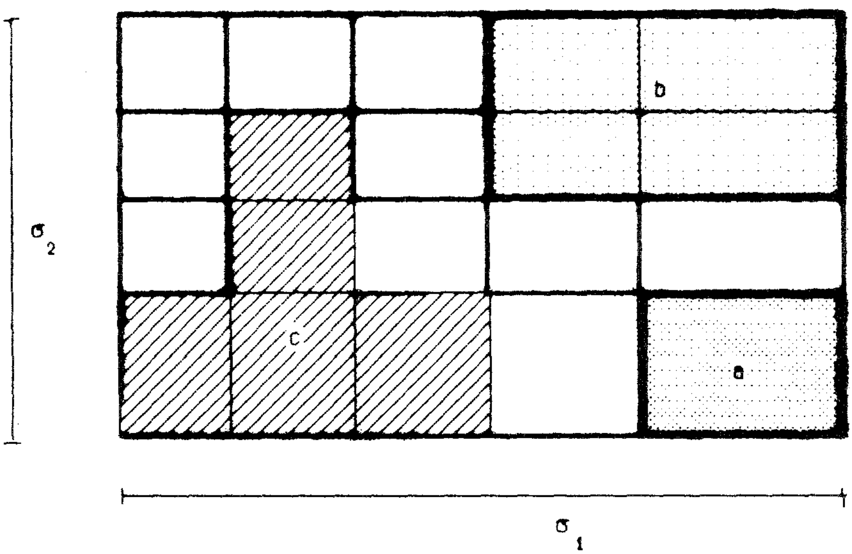


Fig. 1. Two observables $A_1$ and $A_2$ have their spectra $\sigma_1$ and $\sigma_2$ divided into several elementary subsets. The direct product $X = \sigma_1 \times \sigma_2$ is accordingly divided into elementary building blocks (a) corresponding to elementary history predicates and constituting the basis of a Griffiths family (type I). Any union of these building blocks corresponds to a proposition in the family. The set (b) is an example of history predicate not associated with an element of the basis. The set (c) represents a typical general proposition.

union of subsets belonging to the basis), but it might also be the union of some nonelementary history predicates. One shall have to keep this kind of multiple decomposition in mind.

To define the measure of the history predicate $D_0 = C_1 \times \cdots \times C_n$, first consider the case where $\rho = |\psi\rangle\langle\psi|$. Let $F_k$ be the projector associated with $C_k$. Defining

$$|\phi\rangle = F_n(t_n) \cdots F_2(t_2)\, F_1(t_1)\, |\psi\rangle$$

I shall define the measure of $D_0$ as

$$w(D_0) = \langle \phi \mid \phi \rangle \tag{5.4}$$

or equivalently

$$w(D_0) = \mathrm{Tr}\{F_n(t_n) \cdots F_2(t_2)\, F_1(t_1)\, \rho F_1(t_1)\, F_2(t_2) \cdots F_n(t_n)\} \tag{5.5}$$

A more general proposition $D$ corresponding to the union $D_\alpha \cup D_\beta \cup \cdots \cup D_\lambda$ of several history predicates $D_\alpha$, $D_\beta$,..., $D_\lambda$ will be given the measure

$$w(D) = w(D_\alpha) + w(D_\beta) + \cdots + w(D_\lambda) \tag{5.6}$$

**Definition 8.** The measure of any history predicate is defined by Eq. (5.5) and the measure of any proposition by Eq. (5.6), $\rho$ being the state operator.

Notice that, using cyclic invariance of the trace together with the property $F_n^2(t_n) = F_n(t_n)$, Eq. (5.5) can be written as

$$w(D_0) = \mathrm{Tr}[F_{n-1}(t_{n-1}) \cdots F_1(t_1)\, \rho F_1(t_1) \cdots F_{n-1}(t_{n-1})\, F_n(t_n)] \tag{5.7}$$

or, still more explicitly,

$$\begin{aligned} w(D_0) = \mathrm{Tr}[&F_{n-1}\, U(t_{n-1} - t_{n-2}) \cdots F_1\, U(t_1)\, \rho U^{-1}(t_1) \\ &\times F_1 \cdots U^{-1}(t_{n-1} - t_{n-2})\, F_{n-1} \\ &\times U^{-1}(t_n - t_{n-1})\, F_n\, U(t_n - t_{n-1})] \end{aligned} \tag{5.8}$$

A simple way to justify the nontrivial definition (5.5) is the following one: Use the Feynman path integration method,[12] which is known to be equivalent to the Schrödinger equation, i.e., to Rule 3. Then, if the projectors $F_1 \cdots F_n$ refer only to position or momentum operators, the matrix element

$$\langle x_0, 0|\, F_1(t_1) \cdots F_n(t_n)\, |x_1 t_f\rangle \qquad (t_f > t_n)$$

is obtained from the Feynman path integral

$$\langle x_0, 0 \mid x, t_f \rangle = \int_0^{t_f} \prod \left[ dx(t)\, dp(t)/2\pi \right] \exp\left[ i \int_0^{t_f} (p\, dx - H\, dt) \right] \quad (5.9)$$

by restricting the integration over $x(t_1)$ or $p(t_1)$ to a finite set, together with similar restrictions for the integrations at times $t_2 \cdots t_n$. Then, in the case where $\rho = |x_0, 0\rangle\langle x_0, 0|$, expression (5.5) is nothing but

$$\int dx\, |\langle x_0, 0 \mid x, t_f \rangle'|^2$$

where the prime indicates the truncation in the intermediate path integrations. From this point of view, the definition (5.5) becomes quite natural. (Similar quantities also occur in the standard theory.[13]) Obviously, satisfying the axioms of probability calculus in such a formula raises a problem, since Eq. (5.3) requires the additivity of probabilities, whereas one would find here an addition of amplitudes.

### 5.3. Consistency Conditions

Definition 8 should be consistent with the assumptions of probability calculus. More precisely:

(A) The definition should define a unique measure: Given a nonelementary history predicate, one can define its measure in at least two different ways, either directly by Eq. (5.5) or, using a decomposition in elementary history predicates, by Eq. (5.6). More generally, any decomposition of a general set $D = D_\alpha \cup D_\beta \cup \cdots \cup D_\lambda$ into a sum of history predicates should always give the same result.

(B) The axioms (5.1)–(5.3) should be satisfied.

It will be found that these consistency requirements impose strong restrictions upon the families of propositions one can introduce.

The positivity condiction (5.1) is obviously satisfied by Eq. (5.5) in the special case where $\rho = |\psi\rangle\langle\psi|$. In the general case, one can write $\rho = (1/N) \sum_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$, so that positivity is still satisfied. The same is true for the sum in Eq. (5.6).

Unicity is the essential constraint. In fact, it boils down to the following *condition C*: Let $D_0$ be any nonelementary history predicate and $D_\alpha \cup D_\beta \cup \cdots \cup D_\lambda$ its unique decomposition as a sum of elementary history predicates; then the two expressions (5.5) and (5.6) for $w(D_0)$ should coincide. This is obviously a necessary condition for unicity. It is also sufficient because any decomposition of a general proposition into several history predicates will then yield the same measure, namely the sum of the measures of its constituting elementary predicates. It turns out that the axioms of probability calculus are then automatically satisfied: $D \cup D'$ in Eq. (5.3) is the union of the elementary predicates building up, respectively, $D$ and $D'$, so that Eq. (5.3) is trivial. Condition (5.2) also follows because $X$ is the history predicate associated with the set $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$ and each constituting set $\sigma_k$ has for its associated projector the identity operator, so that condition (5.2) reduces to $\mathrm{Tr}\,\rho = 1$.

How Condition C can be expressed as a set of algebraic relations is a problem that is solved in the Appendix. Here I just give a simple illustrative example. Consider the case of two observables $A_1$ and $A_2$ at times $t_1 < t_2$. Their spectra $\sigma_1$ and $\sigma_2$ are respectively divided into two complementary sets $C_1, C_1^*$ and $C_2, C_2^*$ with projectors $F_1, F_1^*, F_2, F_2^*$. The history predicate set $\sigma_1 \times C_2$ is the union of the two elementary history predicate sets $C_1 \times C_2$ and $C_1^* \times C_2$, so that Condition C should be applied to it. This condition reads [omitting the explicit mention of the time, so that I write, for instance, $F_1$ for $F_1(t_1)$]:

$$\mathrm{Tr}\{I\rho IF_2\} = \mathrm{Tr}\{F_1\rho F_1 F_2\} + \mathrm{Tr}\{F_1^*\rho F_1^* F_2\}$$

Replacing $I$ by $E_1 + E_1^*$ and developing the left-hand side of this equation, one gets

$$\mathrm{Tr}\{F_1\rho F_1^* F_2\} + \mathrm{Tr}\{F_1^*\rho F_1 F_2\} = 0$$

Using the Hermiticity of projectors and cyclic invariance of the trace, one finds that the second term is the complex conjugate of the first one, so that the condition becomes

$$\mathrm{Re}\,\mathrm{Tr}(F_1\rho F_1^* F_2) = 0 \tag{5.10}$$

Similarly, for the history predicate $\sigma_1 \times C_2^*$, one gets

$$\mathrm{Re}\,\mathrm{Tr}(F_1\rho F_1^* F_2^*) = 0$$

However, because of the relations $F_2 + F_2^* = I$, $F_1 F_1^* = 0$, and cyclic trace invariance, this is equivalent to Eq. (5.10). The only other nonelementary history predicates are $C_1 \times \sigma_2$ and $C_1^* \times \sigma_2$ and Condition C is trivially satified for them, since it reads, for instance,

$$\mathrm{Tr}(F_1\rho F_1 I) = \mathrm{Tr}(F_1\rho F_1 F_2) + \mathrm{Tr}(F_1\rho F_1 F_2^*)$$

So, in this simple case, there is only one consistency condition, that is, (5.10). It can be written in a different form, which is more typical of the general case. One can write

$$F_1\rho F_1^* + F_1^*\rho F_1 = [F_1, [\rho, F_1^*]]$$

as one sees easily by developing the double commutator and using $E_1 E_1^* = E_1^* E_1 = 0$. Therefore condition (5.10) can be also written in the form

$$\mathrm{Tr}\{[F_1, [\rho, F_1^*]]\, F_2\} = 0 \tag{5.11}$$

Conditions of this type are fundamental in the present approach and they are called consistency conditions.[2]

**Definition 9.** The measure of propositions defined by Eqs. (5.5) and (5.6) must satisfy some *consistency conditions* in order to be unique and to satisfy the mathematical properties of a measure.

Each such condition consists in the vanishing of the trace of a multiple commutator of projectors, as shown in the Appendix, or equivalently as the vanishing of the real part of the trace of a product of projectors in simple cases. Griffiths[2] has shown how to write them all in that last form when the basis of the proposition family is a Griffiths basis, but the conditions he gave are unnecessarily numerous. The smallest number of necessary consistency conditions is given in the Appendix.

A simple physical example of an inconsistent family of propositions is provided by an interference experiment (Griffiths[2,3]; see also II): a screen is pierced by two holes. It is possible to associate two projectors $F_1$ and $F_1'$ with the elementary predicates stating that a particle went through one of them. Let us put another screen behind the first one, divide it into regions $C_{12},..., C_{m2},...$, and consider the family of elementary predicates with projectors $F_{m2}$ stating that the particle hit the second screen in region $C_{m2}$. Then it can be shown that the family of propositions including both families of predicates is inconsistent. Accordingly, this formalism provides a convenient criterion for selecting families of physical propositions that can be considered as logically meaningful. Other examples will be given in II.

Finally, I still give another definition:

**Definition 10.** A physical representation of logic satisfying all the necessary consistency conditions will be called a *consistent quantum representation of logic.*

# 6. IMPLICATION IN QUANTUM MECHANICS

## 6.1. Implication

I have defined history predicates $[S; A_1,..., A_n, C_1,..., C_n, t_1,..., t_n]$ from which I have built up quantum representations of logic with propositions $[D, t_1,..., t_n]$ (omitting some notation) or simply $[D]$. I have found when such a representation is consistent. Now, I want to make a significant step further. After all, logic is not only a description of a system by propositions, but it is essentially a rule for reasoning, i.e., it must allow implication. Implication (here denoted by $\Rightarrow$), i.e., the ability to use a sentence such as, "*If* such proposition, *then* such other proposition," is the building block of science, either in mathematics or in physics.

For the sake of completeness, let us recall what axioms have to be satisfied in logic by a rule of implication:

(i) if $[D] \Rightarrow [D']$ and $[D'] \Rightarrow [D]$, then one must have $[D] = [D']$.

(ii) If $[D] \Rightarrow [D']$ and $[D'] \Rightarrow [D'']$, one must have $[D] \Rightarrow [D'']$.

(iii) $[D] \Rightarrow [D]$.

(iv) If $[D''] \Rightarrow [D]$ and $[D''] \Rightarrow [D']$, one must have $[D''] \Rightarrow [D] \wedge [D']$.

(v) $[D] \Rightarrow [D] \vee [D']$.

(vi) $[D] \wedge [D'] \Rightarrow [D]$.

(vii) If $[D] \Rightarrow [D'']$ and $[D'] \Rightarrow [D'']$, one must have $[D] \vee [D'] \Rightarrow [D'']$.

(viii) If $[D] \Rightarrow [D']$, one must have $[D']^* \Rightarrow [D]^*$.

We shall not use the empty proposition 0 nor the total proposition $I$ such that $0 \Rightarrow [D] \Rightarrow I$ whatever $[D]$. This is known to be a convenient but not a necessary rule for implication.[11]

We have defined the measure $w(D)$ of a proposition. Then, in the case when $w(D) \neq 0$, we can define the conditional measure of proposition $[D']$ given $[D]$ as given by

$$w(D'ID) = w(D \wedge D')/w(D) \tag{6.1}$$

Then we shall use the following theorem in probability theory[14] that is very easily proved by checking the above axioms one by one:

*Theorem.* When $w(D) \neq 0$ and $w(D') \neq 1$, the condition

$$w(D'ID) = 1 \tag{6.2}$$

satisfies the logical axioms defining an implication $D \Rightarrow D'$.

The second condition $w(D') \neq 1$ comes from condition (viii), which is meaningful only when $w(D'^*) \neq 0$. This is the definition of implication I shall use in the consistent representations of logic.

## 6.2. Complementarity

It should be noticed that a given physical system in a given initial state may allow a large number of consistent representations of logic, in fact, as many as one can consistently choose the $A_k$, the $t_k$, and the $D_\mu$. This freedom is what the complementary principle becomes in the present approach.

## 6.3. Operations upon Representations of Logic

These representations can be extended or restricted by the following operations:

**Definition 11.** Let $L_1 = [S, \{D_\mu\}, t_1, ..., t_n]$ be a consistent representation of logic. One can *reduce* it by allowing two different subsets $C_{\mu k}$ and $C_{\mu' k}$ to merge into only one subset of $\sigma_k$. At the limit where all of them merge into the whole of $\sigma_k$, one can as well cease to mention $t_k$ and $A_k$ in the history predicates.

One can also *extend* $L_1$, either by dividing its basis further or by adding other elementary predicates relative to a new variable $A$ with a new time $t$ that may either be prior to $t_1$ although still positive, posterior to $t_n$, or intermediate between $t_1$ and $t_n$. The necessary additional consistency conditions should of course be valid.

These two techniques make it possible to simplify a discussion or to enrich it. For instance, the propositions relating to a subsystem $\Sigma$ of $S$ can be extended to propositions involving the whole of $S$. The discussion of the measurement process will use such possibilities.

## 6.4. Noncontradiction

I now state an important, although somewhat trivial, theorem showing that, once consistency is satisfied, no contradiction can ever occur.

**Theorem 1.** Let $L_1$ and $L_2$ be two consistent representations of logic, both allowing the two propositions $[D]$ and $[D']$. If one has $[D] \Rightarrow [D']$ in $L_1$, then one also has $[D] \Rightarrow [D']$ in $L_2$.

*Sketch of the Proof.* It is easy to show, although somewhat lengthy in full rigor, that the probabilities $w(D)$, $w(D')$, and $w(D \wedge D')$ only depend upon the propositions and not upon their embedding logic.

One important application of this theorem will consist in proving an implication $[D] \Rightarrow [D']$ by using the smallest representation of logic containing both $[D]$ and $[D']$. Examples will be given in II.

It may be worth giving here an example that was considered by Griffith[2] and later criticized on logical grounds by d'Espagnat.[15] One considers a spin-1/2 system with its two-dimensional Hilbert space. Denote, for instance, by $[x, m, t_1]$ the elementary predicate stating that the spin $x$-component at time $t_1$ is equal to $m/2$ ($m = \pm 1$). Let us take the initial state $\rho = F([x, 1, 0])$, i.e., the initial value of $s_x$ is taken to be 1/2. Also consider two representations of logic: on one hand, $L_1$ containing the elementary predicates $[x, m', t_1]$, $[z, m, t_2]$, and on the other hand, $L_2$ containing $[z, m'', t_1]$, $[z, m, t_2]$. The calculations are straightforward and they have been mostly done by Griffiths, so that I shall not reproduce them. The results are the following:

1. Both $L_1$ and $L_2$ are consistent.
2. In $L_2$, one has $[z, m, t_2] \Rightarrow [z, m, t_1]$.
3. In $L_1$ one cannot say that the initial state $\rho$ (that is, a projector) implies $[x, 1, t_1]$, because $\rho \Rightarrow [x, 1, t_1]$ should be equivalent to $[x, 1, t_1]^* \Rightarrow \rho^*$, but $\rho^*$ (the negation of the initial state) does not exist as a predicate in $L_1$.

The consistent use of implication is the main difference between what is obtained here and Griffiths'discussion. It does not lead to any inconsistency, as d'Espagnat could find in Griffiths' formulation, where it was said that the initial predicate could entail the veracity of $[x, 1, t_1]$. No such statement appears in the present formulation.

### 6.5. Approximate Implications

Finally, I relax the perhaps too tight conditions imposed upon consistency and implications:

**Definition 12.** Given a very small number $\varepsilon$, a representation of logic containing a finite number of propositions is consistent up to order $\varepsilon$ if the real part occurring in the consistency conditions has absolute values smaller than $\varepsilon$. I shall say that $D \Rightarrow D'$ *up to order* $\varepsilon$ if $|w(D' \mid D) - 1| < \varepsilon$.

The first definition is justified by the fact that in such a case, one can quite often slightly change the observables in order to satisfy exactly the consistency conditions. I mention it because Griffiths used such approximations.[2] However, I shall try to avoid it for fear of dealing with "approximate" logic.

The second definition is a foretaste of classical physics as founded upon quantum mechanics: for instance, at least in principle, the earth might suddenly, with a nonzero probability, leave the sun and orbit around Sirius by tunnel effect. I want to state precisely how to neglect such effects when stating that the sun will rise tomorrow.

## 7. CLASSICAL PROPOSITIONS

The logical link between classical physics and quantum physics has always been a stumbling block in the interpretation of quantum mechanics.[7,8,16]

### 7.1. Classical Propositions

The predicates of classical mechanics look at first sight quite different from the ones I have introduced up to now. Typically, a classical elementary predicate would express that the state of a physical system $S$ is represented at time $t_1$ by a point in phase space lying within a given cell $C$ of phase space. My first aim will consist in giving a quantum meaning to such a statement.

In classical physics, conjunction and disjunction correspond to intersection and union of such cells, negation to the complementary set in phase space. Implication is generally defined by the inclusion of one cell into another one, so that the whole setup is quite different from what has been proposed up to this point for quantum mechanics.

My program, however, will be to recover completely this form of classical logic as a result of quantum logic, at least in favorable circumstances. To begin with, I shall associate a projector in Hilbert space with a classical predicate saying that the state of the system is in a cell of phase space.

### 1.2. Phase Space

First, I have to specify what is meant by phase space, from a quantum point of view. Consider once again the case of a particle as a typical example.

The mathematical notion of phase space arises from the quantum notions of Hilbert space and observables in the following way.[17,18] To an observable $A$, one can associate a real function $a(x, p)$ of two variables $(x, p)$ (i.e., a function on phase space) that is called the *symbol* of $A$ and is given by

$$a(x, p) = \int dx'\, dx''\, \delta[x - (x' + x'')/2] \exp[ip(x' - x'')] \langle x'|\, A\, |x''\rangle \qquad (7.1)$$

This leads us to the so-called Weyl calculus, which introduces phase space in a mathematically well-defined way.[17] Conversely, the action of the operator $A$ on a wave function $u(x)$ gives a wave function $v(x)$ that satisfies

$$v(x) = \int \exp[-ip(x - x')]\, a[(x + x')/2, p]\, u(x')\, dx'\, dp/(2\pi) \qquad (7.2)$$

Despite its apparently cumbersome form, this correspondence between operators and functions on phase space is extremely fruitful in the theory of partial differential equations[19] and it can also be used in quantum mechanics.[18,20] The main point is that, when we look for some operator $A$, it is enough to find its so-called $W$ symbol $a(x, p)$. The theory of symbols is called microanalysis and it is a very active part of contemporary mathematics.[19]

I now state what is meant by a macroscopic cell in phase space:

**Definition 13.** A cell in phase space is said to be *a macroscopic cell* if it has the two following properties: (1) Its volume (in units where $h/2\pi = 1$) is large. (2) Its boundary is regular.

Here regularity is a technical mathematical condition. It can be stated precisely as follows: Let $L$ denote a typical size of $C$ for any $x$ direction and $P$ a similar size for any $p$ direction. Then one assumes that the curvature of the boundary in the metric

$$ds^2 = (L^{-2}\, dx^2 + P^{-2}\, dp^2)(1 + L^{-2}x^2 + P^{-2}p^2)$$

is of the order of unity in dimensionless variables $L^{-1}x$, $P^{-1}p$.

The quantum formulation of a classical predicate comes from the following theorem.

**Theorem 2.** To a macroscopic cell, in phase space, one can associate an approximate projecto $\Phi(c)$ and a corresponding state operator $\Phi/\mathrm{Tr}\,\Phi$.

This is a well-known theorem[1,18,21] (see also III). As an example I sketch what is actually an approximate projector when $C$ is taken to be the simple cell in a two-dimensional phase space,[21]

$$x^2 + p^2 < R^2 \tag{7.3}$$

in convenient units, $R$ being a large number. Here I shall not use microlocal analysis, but a wavelet expansion[22] that is a special case of it. Define the functions

$$g^{(q,p)}(x) = \exp[(-ipq/2) + ipx]\; g(x - q)$$

where $g(x)$ is the Gaussian $\pi^{-1/4}\exp(-x^2/2)$. Define an operator $\Phi$ acting on a wave function $u$ by

$$\Phi u = (2\pi)^{-1} \int_C dp\, dq\; g^{(p,q)} < g^{(p,q)}\, |u\rangle \tag{7.4}$$

When $C$ is the disk (7.3), it is easily shown that $\Phi$ commutes with the harmonic oscillator Hamiltonian $h = 1/2(x^2 + p^2)$. Therefore, its eigenvectors are the Hermite functions $H_n(x)$. The corresponding eigenvalues $\lambda_n$ turn out to be given by

$$\lambda_n = (n!)^{-1} \int_0^{R^2/2} dt\, t^n \exp(-t)$$

It is easily found that $\lambda_n$ is positive, nearly equal to 1 for $n < R^2/2$, very small for $n > R^2/2$, with a transition region of size $R$ around $R^2/2$, where $\lambda_n$ gradually steps down from 1 to 0. This is what I call an approximate projector. When entering a trace as, for instance, in a predicate measure, the relatively small number of eigenvalues lying between 1 and 0 will not cause the results to differ appreciably from those obtained from a strict projector.

The general case, in several dimensions for any macroscopic cell $C$, is of course more complicated. One can prove that the operator $\delta\Phi = \Phi^2(C) - \Phi(C)$ has a norm of the order of 1 and a trace of the order of $\Delta(\mathrm{Tr}\,\Phi(C))$, where

$$\Delta = (\hbar/LP)^{1/3}$$

I shall treat $\Phi(C)$ as if it were an exact projector.

It might be mentioned that such an approximate projector is not unique. For instance, one might have used Gaussians with an uncertainty in $x$ different from 1, or use other kinds of functions, or even not use wavelet expansion and work with a more general microlocal construction. The relative errors are always of the same order.[18]

Different macroscopic cells have the following important property (see III).

**Theorem 3.** Let $C$ and $C'$ be any two fixed macroscopic cells in phase space and let the Planck constant $h$ tend to zero; then the commutator $[\Phi(C), \Phi(C')]$ tends to zero in norm. More precisely, it has a norm smaller than 1 and a trace of order $\Delta \inf(\mathrm{Tr}\,\Phi(C), \mathrm{Tr}\,\Phi(C'))$ in absolute value, with $\Delta$ as above.

## 7.3. Classical Logic as a Limit

I now come to the relation between classical logic and consistent quantum representations of logic. The first step will again be a mathematical definition.

**Definition 14.** A Hamiltonian $H$ is said to be *regular* during a time $T$ for a macroscopic cell $C$ if its classical version [i.e., its symbol $h(x, p)$], when used to generate a classical motion via Hamilton's equations, preserves the macroscopic character of $C$ during the time interval $[0, T]$.

For example, for mixing Hamiltonians (e.g., Kolmogorov's $K$-flows[23]), $T$ is a rather short but macroscopic time. For the harmonic oscillator, it is infinite; for a free particle, it is only limited by the spreading of wave packets $(ht/m)^{1/2}$, so that $T$ can be taken very large when $C$ is large enough.

Then, one has the following theorem that in principle should essentially reduce quantum physics to classical physics (including both classical mechanics and classical logic) when one is dealing with macroscopic cells and regular Hamiltonians.

**Theorem 4.** Assume that a physical system is initially in a state defined by a finite macroscopic cell $C_0$, i.e., by a state operator $\rho = \Phi(C_0)/\mathrm{Tr}[\Phi(C_0)]$. Let us also assume that the Hamiltonian is regular for this cell of phase space during a time $T$. Let us call $C_1, \ldots, C_n$ the cells that are the transforms of $C_0$ by classical motion for a finite sequence of times $t_1 < \cdots < t_n < T$. Let us introduce the smallest quantum representation of logic $L$ containing the corresponding predicates and their negations.

Then, when the Planck constant tends to zero, one has the following limiting results:

1. $L$ is consistent.
2. $L$ is a sublogic (i.e., a Boolean subalgebra) of classical logic.
3. Denoting by $[C_k]$ the proposition stating that the system is in cell $C_k$ at time $t_k$, one has $[C_j] \Rightarrow [C_k]$ whatever $t_j$ and $t_k$, both positive and smaller than $T$.

The complete proof of this theorem has only been obtained recently and it needs all the powerful machinery of microlocal analysis. It will be left for a more mathematical publication. I only add the following precisions: Let $L$ and $P$ be, as above, a typical size of $C$ in the $x$ directions and in the $p$ directions. Let $\Delta = (\hbar/LP)^{1/3}$. Then, if $\Phi(C, t_k) = U(t_k)\, \Phi(C_0)\, U^{-1}(t_k)$, putting $\delta\Phi = \Phi(C, t_k) - \Phi(C_k)$, it can be shown that the norm of $\delta\Phi$ is of the order of 1 and its trace of the order of $\Delta(\mathrm{Tr}\, \Phi(C_0))$. The above implications are valid up to an error of order $\Delta$.

It should be noticed that what has been called here a macroscopic cell does not assume that the physical system is itself macroscopic.

## 7.4. Macroscopic Objects

The case of macroscopic objects is far from simple. To specify that one deals with a macroscopic object involves two conditions[8]: (i) the physical system has a large number of degrees of freedom; (ii) it is in a state that can be reliably described by some collective variables. Generally, only the collective variables are given in macroscopic cells of their relevant phase space, whereas the state of the remaining microscopic variables is still described by an ordinary state operator and often in practice by a density operator. The whole subject of irreversible thermodynamics lies behind such a description. However, it is genally granted that such a description is possible in principle and many specific examples are known in the form of models.

I cannot say more, except to believe, as everybody does, that once this is done, the averaged collective Hamiltonian can in many cases be considered as regular, in the sense of Definition 15, so that Theorem 4 may be expected to hold for the cells describing the collective variables. I say that such a system is *collectively regular*. There is a close relation between collectively regular macroscopic systems and systems having a classical dynamical limit that behaves in a deterministic way, at least for a finite length of time.

### 7.5. Potential Facts

Assuming this, I now come to an essential point: How to exhibit facts from an intrinsically probabilistic theory.

I first define this basic notion for a system that remains in a macroscopic cell. The simplest example is given by a particle that is initially in a state defined by the projector $\Phi(C_0)$ of a macroscopic cell.

**Definition 15.** Let an isolated individual physical system $S$ be initially in a state with the state projector $\Phi(C_0)$ associated with a macroscopic cell in phase space $C_0$. Assume that the Hamiltonian is regular for $C_0$ during time $T$ and denote by $C_t$ the cell transformed from $C_0$ at time $t$ under classical motion. Then the occurrence of $S$ in the cells $C_t$ at time $t$ $(t < T)$ is a *potential fact* because one has $[C_t] \Rightarrow [C_{t'}]$ and $[C_{t'}] \Rightarrow [C_t]$ whatever $t$ and $t'$ in the time interval $(0, T)$.

Stated in words, a potential fact is a chain of macroscopic events, not necessarily concerning a macroscopic object, that leave a memory allowing a reconstruction of the past and a deterministic prediction of the future during a finite time interval. Their existence is a consequence of Theorem 4. Note that I say "potential fact" rather than just "fact" because, up to now, one can only consider such a "fact" as being a chain of propositions.

An interesting example is provided by Wigner's beam recombination experiment[24,7]: A spin-1/2 particle is initially in a pure spin state $s_x = 1/2$, whereas its initial position–momentum state at time zero is described by a macroscopic cell $C_0$ corresponding to an initial beam going along the $y$ axis. This beam goes through a Stern–Gerlach apparatus $S_z$ able to separate the two spin states $s_z = \pm 1/2$ that may give rise to two well-separated beams, where, for instance, the beam followed by a particle with $s_z = 1/2$ has a positive $z$ coordinate. Call $C_{z^+}$ a typical cell in phase space describing the classical macroscopic situation of the upper beam at some time $t$. A similar cell for the other beam will be denoted by $C_{z^-}$. An external magnetic field acting upon the particle magnetic moment can recombine the two beams into a single one following again the $y$ axis. Call $C'$ a typical cell describing this recombined beam. Finally, this beam crosses another Stern–Gerlach apparatus $S_x$ able to separate the two spin states $s_x = \pm 1/2$.

The occurrence of the particle in the cells $C_{z^+,t}$ is, according to Definition 15, a potential fact as long as the two beams $C_{z^+,t}$ and $C_{z^-,t}$ do not recombine. The point is that, in that case and assuming the Hamiltonian to be regular, one has $[C_{z^+,t}] \Rightarrow [C_{z'^+,t'}]$ and $[C_{z'^+,t'}] \Rightarrow [C_{z^+,t}]$ as a result of Theorem 4, $t$ and $t'$ denoting two times posterior to the crossing of the apparatus $S_z$ and prior to the recombination of the beams. In order to prove these implications, one must use $[C_{z^+,t}] \Rightarrow [s_z = 1/2, t]$, which determines the trajectory in the exterior magnetic field.

This chain of potential facts breaks up after recombination. Indeed, one has $[C_{z^+,t}] \Rightarrow [C'_{t'}] \wedge [s_z = 1/2, t']$, so that $[C_{z^+,t}] \Rightarrow [C'_{t'}]$, but one cannot have the converse implication $[C'_{t'}] \Rightarrow [C_{z^+,t}]$, only $[C'_{t'}] \Rightarrow [C_{z^+,t}] \vee [C_{z^-,t}]$, so that the conditions of Definition 15 are not satisfied.

It should benoticed that I have used here an absolute time and that relativity has not been taken into account. So, the notion of potential fact as given in Definition 15 is restricted to nonrelativistic physics. This is a severe limitation, but it is balanced by the advantage of being a rather precise notion.

Things become more delicate when one deals with an incompletely specified system, such as a macroscopic object. The collective variables may be insufficient to predict the evolution of the system and many things could remain hidden in the microscopic variables. An example is Schrödinger's cat experiment, where we take the macroscopic object to be the box containing the cat *and* the radioactive source. Of course, there are many macroscopic systems that cannot keep a trace, as may happen, for instance, in a turbulent flow. One must in general restrict the quest for facts to a reliable description by a small number of classically deterministic collective variables, in practice to what I called a classically regular system.

It should be mentioned that an intrinsic definition of collective variables, using only the quantum dynamical framework for a given state, remains one of the few important problems in theoretical physics that has practically not been touched upon.

The objects allowing potential facts may be said to keep a memory. A memory (e.g., the track of a particle in a crystal) is described by a classical proposition relative to some collective variables (often quite many of them, but very few when compared to the number of degrees of freedom) from which one can logically reconstruct the past (e.g., perform an autopsy of Schrödinger's cat or take its temperature). However, this logic is the classical one, so that one must understand what is the status of logic in a world obeying quantum mechanics. To do so, one must assert more precisely the physical status of the consistent quantum representations of logic.

## 8. AN INTERPRETATIVE AXIOM

I shall try to put some order into difficult questions by stating a new bold rule for founding physics:

**Rule 5.** The consistent quantum representations of logic provide all the possible descriptions of a physical system $S$ in a given initial state and they also provide the correct physical means for reasoning about them.

It should be stressed that Rule 5 does not assume *a priori* the use of classical logic. On the contrary, classical logic as we apply it everyday is considered here as a consequence of Rule 5 using Theorem 4.

Furthermore, it should be stressed that Rule 5 provides the means of replacing any talkative argument about an experimental situation in quantum mechanics by a calculation. Given any tentative or intuitive reasoning about such a question or, let us say, a preformalized argument, one can check whether all its statements enter into a consistent representation of logic and whether all its logical links are justified by an implication. Such a procedure may be considered as providing a quantum calculus of propositions that probably could be used to eliminate many spurious discussions from the field. Examples will be given in II.

### 8.1. The Existence of Facts

Using Rule 5, one can derive from Theorem 4 that a system initially defined by a state projector $\Phi(C)$ obeys classical logic if its Hamiltonian is regular. I shall also assume that there exists macroscopic systems, so that Theorem 4 applies to their collective variables. If it happens that their initial state allows them to be collectively regular, then these objects obey classical dynamics and their description obeys classical logic.

Whether such collectively regular systems may exist theoretically and how one should explicitly define the collective variables are questions leading far beyond the present limits of theoretical investigation. It is enough to notice that nothing known tells us that the existence of collectively regular systems is impossible and that we see them occurring everyday, so the answer to these questions can be reasonably assumed to be positive, at least in principle.

Restricting attention to such collectively regular objects (e.g., a voltmeter), they can be described by classical logic, here understood as a special limiting case of consistent quantum logics. Then the sentence describing potential facts in Definition 15 means that the state projector of the object $\Phi(C_0)$ is an initial datum resulting from its previous history. Probably, many systems tend spontaneously to generate a chain of potential facts, even when starting from a state that does not belong to a family of potential facts: think, for instance, of a turbulent liquid, dissipating velocity, cooling down by radiation, and turning into a crystal. The collective variables describing the shape and the kinematics of the crystal are then able to describe potential facts. Of course, turning these common sense considerations into a theoretical treatment involves a good understanding of irreversible thermodynamics.

My conclusion will not be that potential facts must exist in quantum mechanics, but that nothing known forbids them to exist. Their existence will be taken as granted.

### 8.2. What is True?

Finally, in accordance with Heisenberg,[25] one can restrict the word "true" in an actual sense to facts. Here one does not speak of potential facts but of situations actually occurring at the macroscopic level, as one can see them and as the theory in principle describes them, at least by propositions. Among macroscopic objects, such facts may exist as a matter of principle, according to the preceding considerations. The link with experiment is to provoke and to observe them.

From some facts that are actually true, one can logically derive some quantum propositions by implication. For instance, it will be shown in II that, when the decay of a particle is a fact and when one of its decay products $Q$ is found to be actually in a small region of space by being detected by a counter some distance away, then one can tell by implication that $Q$ was in some other region on the way at a previous time $t$. Shall we say, however, that this implied statement is also true? It would be logically dangerous because another consistent representation of logic also allows us to derive by implication that the momentum of $Q$ is in a well-defined region of momentum space with the same choice of time. Both statements cannot be said logically to be true, because they would then have to be true together. Here we meet once again the problem of complementarity as when discussing light either as an electromagnetic field or as made up of photons.

The occurrence of complementarity through the variety of consistent quantum representations of logic should therefore make us careful when using the word "true." Accordingly, I propose the following definition:

**Definition 16.** Any proposition that can be derived by implication from actual facts is said to be *a reliable statement.*

Reliable statements can never meet contradiction as a result of Theorem 1: no other representation of logic including the corresponding proposition and the facts will ever lead to a contradiction.

## 9. THE THEORY OF MEASUREMENT

I now discuss the theory of measurement as it occurs in the present approach. First, since we have the notion of fact at our disposal, it is possible to describe the result of an individual measurement within the framework of quantum mechanics without referring explicitly to classical physics and classical logic. Accordingly, I shall treat both the measured system and the measuring apparatus as being quantum systems and restrict considerations to an actual measurement for which the result given by the apparatus is an actual fact.

### 9.1. Description and Notation

Here I fix the notations. The measurement is done on a quantum system $Q$ (e.g., a particle) that is described by a Hilbert space $\mathscr{H}^Q$ and Hamiltonian $H^Q$. It is initially in a state described by the state operator $\rho^Q$.

The measurement is realized by using a macroscopic apparatus $M$ with Hilbert space $\mathscr{H}^M$ and Hamiltonian $H^M$. As a measuring aparatus, it has the following property: one of its degrees of freedom describes, for instance, a needle $N$ that can have its position in well-separated intervals $J_0, J_1,..., J_n,...$ of the real axis. To these intervals are associated projectors $E_0^M,..., E_n^M,...$ on orthogonal subspaces of $\mathscr{H}^M$. Let us call $N$ the apparatus observable describing the position of the needle. Initially, the needle is in position zero (i.e., in $J_0$) and I shall simply take for the corresponding state operator $\rho^M = E_0^M/\mathrm{Tr}\, E_0^M$.

Let us consider an observable $A$ associated with the quantum system $Q$. To begin with, consider the simple case where $A$ has a nondegenerate discrete spectrum $\{a_1, a_2,..., a_n,...\}$.

The complete Hilbert space of the system $Q+M$ is the tensor product $\mathscr{H}=\mathscr{H}^Q \otimes \mathscr{H}^M$; the complete Hamiltonian will be written as

$$H = H^Q + H^M + H_{\text{int}} = H_0 + H_{\text{int}}$$

where $H_{\text{int}}$ represents the interaction between $Q$ and $M$. The initial state operator is taken to be $\rho^Q \otimes \rho^M$.

## 9.2. Characterizing a Measuring Apparatus

$M$ is supposed to be a measuring apparatus for the quantum dynamical variable $A$. This property can be expressed as a characteristic of the interaction between $Q$ and $M$: Denote by $|a_n\rangle$ the normed eigenstates of $A$ in $\mathscr{H}^Q$. Let $|nb\rangle$ represent the orthonormal states of the apparatus with the needle being in interval $J_n$, $b$ standing for all the other quantum numbers. The states in the overall Hilbert space $\mathscr{H}$ can then be expanded along the orthogonal basis $|a_p\rangle \otimes |nb\rangle = |nba_p\rangle$.

Assume that the interaction takes place during a short time interval between time $t-\varepsilon$ and $t$. The evolution operators $U(t') = \exp(-iHt')$ for the overall system and $U_0(t') = \exp(-iH_0 t')$ for a noninteracting system are therefore related by

$$\begin{aligned} U(t') &= U_0(t') && \text{for} \quad t' < t-\varepsilon \\ U(t') &= U_0(t'-t)\, S U_0(t-\varepsilon) && \text{for} \quad t' > t \end{aligned} \tag{9.1}$$

Here $S$ plays the role of the $S$-matrix (collision matrix) between $Q$ and $M$, the collision occurring between times $t-\varepsilon$ and $t$.

The fact that $M$ is a measuring apparatus can be expressed as a property of the operator $S$. It is essentially a partial diagonalization of $S$ in the variables $A$ and $N$ such that

$$S\,|0, b, a_n\rangle = \sum_{b'} \Theta_{bb'}^{(n)}\, |n, b', a_n\rangle \tag{9.2}$$

The fact that $a_n$ appears in the left-hand side of Eq. (9.2) means that the observable $A$ is not changed during the interaction. This is characteristic of a so-called measurement of the first kind.[26] Note also that the initial state of the apparatus does not appear in the left-hand side, meaning that the measurement is perfectly efficient. Of course, the essential property is that one finds the final state of $M$ with the needle in position $n$ $(N=n)$ when the system $Q$ is initially in the state $A=a_n$.

### 9.3. Measurement as Part of a History

I shall now recast measurement theory in the framework of the present proposition calculus. It will be instructive to consider not only the measurement as such, but also to find how a true fact that is registered by the measuring apparatus at time $t$ enters a consistent logical description of the measured system $Q$. Accordingly, I also want to introduce elementary predicates describing $Q$ before interacton, when it is still an isolated system, and also after the interaction when it is once again isolated.

In order to be sufficiently general without introducing too heavy notations, I restrict consideration to the following case: At a time $t_1$ before $t-\varepsilon$, I consider the elementary predicate $[Q, A_1, C_1, t_1]$ for some observable $A_1$ being in some range $C_1$, as well as its negation. Its projector will be denoted by $F_1(t_1)$. Let us also consider the elementary history predicate $[Q, A_2, A_3, C_2, C_3, t_2, t_3]$ with $t<t_2<t_3$. The projector associated with the elementary predicate $[Q, A_2, C_2, t_2]$ (resp. $[Q, A_3, C_3, t_3]$) will be denoted by $G_2(t_2)$ [resp. $G_3(t_3)$]. I also introduce the elementary predicates that express the value of the measured quantity $A$ before and after measurement, i.e., $[Q, A, a_n, t-\varepsilon]$ and $[Q, A, a_n, t]$.

Measurement will be described by the elementary predicate $[N, n, t]$ expressing the situation of the measuring apparatus just after interaction, or, in words, "At time $t$, the needle is in position $n$." It is associated with a projector $E_n(t)$. All the projectors can be understood either in the relevant Hilbert space or in the overall Hilbert space, but for a trivial tensor multiplication by a unit matrix.

In order to define a representation of logic containing all these predicates, I use the basic set

$$X=\sigma_{A1}\times\sigma_A\times\sigma_N\times\sigma_A\times\sigma_{A2}\times\sigma_{A3} \tag{9.3}$$

I shall write the basis of the proposition lattice $B$ in a convenient short-hand notation where the same letter denotes a predicate, a projector, or their complements: let $f_1$, for instance, denote either the subset $C_1$, the elementary predicate $[Q, A_1, C_1, t_1]$, or the projector $F_1(t_1)$ associated with it or their negation or complement $C_1^*$, $[Q, A_1, C_1^*, t_1]$, or $F_1^*(t_1)$. Use a similar short-hand notation $g_2$ and $g_3$ for the similar elementary predicates at times later than $t$. Use similarly simplified notations $d$ and $h$ for the elementary predicates $[Q, A, a_n, t-\varepsilon]$ and $[Q, A, a_n, t]$ or their negations. Finally, denote simply by $e_n$ the interval $J_n$, the predicate $[M, N, J_n, t]$, or the projector $E_n(t)$, and by $e_n^*$ the set $J_n^*$, the predicate $[M, N, J_n^*, t]$, or $E_n^*(t)$. The basis of the lattice $B$ will then be taken to consist of sets having the form

$$(f_1\times d\times h\times e_n\times g_2\times g_3) \qquad \text{or} \qquad (\sigma_{A1}\times\sigma_A\times\sigma_A\times e_n^*\times\sigma_{A2}\times\sigma_{A3})$$

With that choice, $B$ is the smallest family of events that contains all the relevant predicates. As usual, the corresponding measures are traces of projector products. The corresponding representation of logic will be denoted in the following by $L$.

## 9.4. Measurement and Logical Consistency

Let us now see whether this representation of logic $L$ is consistent There are two kinds of compatibility conditions. The first type involves only the projector $E_n(t)$. The subfamily of the other projectors being a Griffiths' family (type I), the consistency conditions where $E_n$ enters will be given by Theorem 1 and they can be written as

$$\begin{aligned}\operatorname{Re}\operatorname{Tr}\{&g_2(t_2)\,h(t)\,E_n(t)\,d(t-\varepsilon)\,f_1(t_1)\,\rho f'_1(t_1)\\&\times d'(t-\varepsilon)\,E_n(t)\,h'(t)\,g'_2(t_2)\,g_3(t_3)\}=0\end{aligned}\tag{9.4}$$

where $f_1$, for instance, can be either $F_1$ or $F_1^*$, and so on. In the compatibility conditions only one element of the set $(f_1, d, h, g_2)$ differs from the corresponding element of the set $(f'_1, d', h', g'_2)$ and they are complementary.

The second type of compatibility condition bears only upon the apparatus reading and it reduces to one condition, namely

$$\operatorname{Re}\operatorname{Tr}[E_n(t)\,\rho E_n^*(t)]=0$$

which is trivially satisfied.

Using Eq. (9.1), one can write the trace in Eq. (9.4) as

$$\begin{aligned}\operatorname{Tr}\{&U_0(t_3-t_2)\,g_2\,U_0(t_2-t)\,E_nhSU_0(t-\varepsilon-t_1)\,f_1\\&\times U_0(t_1)\,\rho U_0^{-1}(t_1)\,f'_1S^\dagger\\&\times U_0^{-1}(t-\varepsilon-t_1)\,h'E_nU_0^{-1}(t_2-t)\,g'_2U_0^{-1}(t_3-t_2)\,g_3\}\end{aligned}\tag{9.5}$$

One can simplify this expression by noticing first that $U_0(t')$ reduces to $U_0^Q(t')\otimes U_0^M(t')$ for $0<t'<t-\varepsilon$, then that $U_0^M(t-\varepsilon)\,\rho^M U_0^M(t-\varepsilon)^{-1}$ is proportional to $E_0^M(t-\varepsilon)$; furthermore, the diagonal property (9.2) of $S$ tells us that $E_n(t)\,h(t)\,Sd(t-\varepsilon)\,E_0(t-\varepsilon)$ is equal to zero, except when $h=d=|a_n\rangle\langle a_n|$. Therefore, the trace (9.5) becomes, after a little algebra,

$$\begin{aligned}&\operatorname{Tr}_Q[g_2^Q(t_2)\,e_n^Q(t)\,g_2'^Q(t_2)\,g_3^Q(t_3)]\\&\quad\times\operatorname{Tr}_Q[f_1^Q(t_1)\,\rho^Q f'(t_1)\,e_n^Q(t-\varepsilon)]\left(\sum_{bcd}\Theta_{bc}^{(n)}\rho_{cd}^M\Theta_{db}^{(n)*}\right)\end{aligned}\tag{9.6}$$

This remarkable factorization works for the consistency conditions and for the measures of predicates. It has several important consequences.

## 9.5. Consistency Conditions

The first consequence of (9.6) is its effect on consistency conditions. It will be convenient to introduce two different representations of logic $L_{\text{before}}$ and $L_{\text{after}}$:

$L_{\text{before}}$: initial state operator at time zero $\rho$, elementary predicates $f_1$ at time $t_1$ and $d$ at time $t-\varepsilon$.

$L_{\text{after}}$: initial state operator $|a_n\rangle\langle a_n|$ at time $t$, predicates $g_2$ and $g_3$ at later times.

Both are relative only to the quantum system $Q$.

The consistency conditions for $L$ require that the real part of (9.6) vanishes. Now it turns out that among the first two factors in Eq. (9.6), one of them is always a measure, say, e.g., for $L_{\text{before}}$, and therefore real; the other is a trace occurring in a consistency condition (in that case for $L_{\text{after}}$). So a necessary and sufficient condition for $L$ to be consistent is therefore that both $L_{\text{before}}$ and $L_{\text{after}}$ be consistent.

So, one is led to conditions involving only on one hand what happens to $Q$ prior to the measurement and up to the beginning of measurement and, on the other hand, what happens to $Q$ after the measurement as if the initial tate operator were $|a_n\rangle\langle a_n|$ *at the time immediately following measurement.* It is enough that each of these impler representations of logic be consistent with the above restrictions for the larger representation of logic to be also consistent.

## 9.6. The Logical Implications Following a Measurement

The factorization that has been found for the traces entering the compatibility conditions holds as well for the probabilities. Assuming $L$ to be consistent, one then gets as an immediate consequence of Eq. (9.2) the following simple but essential implications holding in $L$:

$$\begin{aligned} [M, N, n, t] &\Rightarrow [Q, A, \{a_n\}, t-\varepsilon] \\ [M, N, n, t] &\Rightarrow [Q, A, \{a_n\}, t] \end{aligned} \tag{9.7}$$

When the left-hand side of (9.7) describes an actual fact, the right-hand sides become reliable statements according to Definition 16.

One can therefore consider in general the result of a measurement, when expressed as a predicate over the measured system, to be a reliable statement. In the same way, the description of the initial state operator after preparation, as given by Rule 2, is also a reliable statement.

These results can be summed up by the following.

**Theorem 5.** Consider the measurement of an observable $A$ having a nondegenerate discrete spectrum by a measuring apparatus of type I.[26] Assume that the state of the measuring apparatus after measurement belongs to a chain of potential facts. Then, in any consistent representation of logic containing the result of the measurement as a proposition, one must have: as a reliable statement, that the value of $A$ before measurement is one well-defined eigenvalue $a_n$; and any such consistent representation of logic making it possible to describe the measured system once it is again isolated after time $t$ must use the initial state $|a_n t\rangle\langle a_n t|$ at time $t$.

This is of course wave packet reduction, which appears as a theorem in the framework provided by Rule 5. Its logical status has become quite definite and the apparatus did not even have to be described by classical logic.

## 9.7. A Possible Experimental Test

In the case of Wigner's recombination experiment, as described in Section 7, this analysis predicts that the second Stern–Gerlach apparatus $S_x$, if followed by a counter detecting the particle, will give the same result as if the first Stern–Gerlach apparatus $S_z$ had not been there. This is the only place where I could find a difference between the experimental predictions of the present theory and those of standard quantum mechanics. Unfortunately, standard quantum mechanics is not entirely predictive in this case,[24,7] as opposed to the present theory.

## 9.8. Generalizations

It is easy to generalize the above results along two directions. I shall not detail the proofs, which follow exactly the same lines.

1. Assume that the measuring apparatus does not give a precise nondegenerate eigenvalue $a_n$ for $A$ from $N$ but just a set $\{a_{n1},..., a_{np}\}$ of eigenvalues that may be degenerate. Then one may consider the proposition $[M, N, n_1, t] \vee \cdots \vee [M, N, n_p, t] = \Pi$. One has

$$\begin{aligned} &\Pi \Rightarrow [Q, A, \{a_{n1},..., a_{np}\}, t-\varepsilon] \\ &\Pi \Rightarrow [Q, A, \{a_{n1},..., a_{np}\}, t] \end{aligned} \tag{9.8}$$

Furthermore, taking the case where $\rho^Q = |\psi\rangle\langle\psi|$, introducing the subspace $P$ of $\mathcal{H}^Q$ that is spanned by $|a_{n1}\rangle,..., |a_{np}\rangle$, the normed vector $|\psi_{1(t)}\rangle$

in $P$ along which lies the projection of $|\psi(t)\rangle$, as well as the predicate $\pi$ that is associated with the projector $|\psi_{1(t)}\rangle\langle\psi_{1(t)}|$, one finds that $\Pi \Rightarrow \pi(t-\varepsilon)$ and $\Pi \Rightarrow \pi(t)$, so that one recovers the well-known formulation of wave packet reduction in such a case.[13] The same result presumably holds when one is dealing with an interval in a continuous spectrum rather than with a set of discrete eigenvalues.

2. The case of a measurement of type II, where the state of $Q$ is not the same before and after the measurement, is also straightforward. Replacing in Eq. (9.2) the state $|n, b', a_n, t\rangle$ by $|n, b', t\rangle \otimes |\psi_n, t\rangle$, where $|\psi_n, t\rangle$ is the state of $Q$ after measurement if initially it is in the state $|a_n, t-\varepsilon\rangle$, one just uses $\rho^Q(t) = |\psi_n, t\rangle\langle\psi_n, t|$ for the initial state operator after measurement.

## 9.9. Not All Observables Are Measurable

The status of observables that can be actually measured as distinguished from an arbitrary self-adjoint operator becomes clearer: There is no reason to assume that the large family of self-adjoint operators in an infinite Hilbert space is made up of quantities that can in principle be measured. It appears that only a few dynamical variables can be actually measured by an actual apparatus. Nevertheless, many others can just as well enter propositions in consistent representations of logic, i.e., they have a logical meaning, even if not an experimental one.

The reason for the specification of the state operator in Rule 2 may also become clearer by now. The preparation of a state results most frequently from some fact that is associated with a finite-rank projector. Not every state operator can be realized, for the same reason that not every observable can be measured. This restriction is due to the limited number of typically different measuring devices.

## 9.10. The Probability Rule

It should be noticed that I did not need to assume that some eigenvalue of the observable $A$ had to be the result of an individual measurement. This is a necessary consequence of the theory.

Finally, the measure of the proposition stating that the macroscopic observable $N$ (needle position) has the value $n$ is easily shown to be given directly by a trace involving only the measured system $Q$, namely

$$\mathrm{Tr}[\rho^Q E_n^Q(t)] = \langle\psi|\, E_n^Q(t)\, |\psi\rangle = |\langle\psi \mid a_n\rangle|^2 \tag{9.9}$$

if $\rho Q = |\psi\rangle\langle\psi|$. The independence of this result from the peculiarities of the measuring apparatus gives it an intrinsic meaning.

Using Rule 4, one can in principle describe a series of independent individual trials, so that it becomes at last possible to give a physical meaning to the mathematical measures as *bona fide* probabilities:

**Rule 6.** The mathematical measure associated with a given result for a measurement is the probability of obtaining this result in a series of independent trials.

## 10. IDENTICAL PARTICLES

Nothing has been said about identical particles and how to deal with Pauli's exclusion principle in particular. This question should be worth a special investigation and it has not yet been considered in full detail. Its most difficult aspect is its apparently "holistic" character or, as stated by Margenau,[28,8] should we have to antisymmetrize with respect to all the electrons in the universe? Here, I just make two elementary remarks concerning, on one hand, two electrons in the same macroscopic cell of phase space and, on the other hand, measurement on a two-electron system, just to show that the main results are not invalidated.

### 10.1. Two Electrons in a Macroscopic Cell

Let us consider the predicate stating that two electrons are in the same macroscopic cell of phase space $C$. We want to find the associated projector. I shall only treat the case of one degree of freedom. A spin-1/2 state with $s_z = \alpha$ will be denoted by $|\alpha\rangle$ in a two-dimensional Hilbert space.

It is possible to generalize Eq. (7.4) giving a projector in the following way: Define some kets for two electrons as given by

$$G_0^{(q,q',p,p')}(x, x', \alpha, \alpha') = g^{(q,p)}(x)\, g^{(q',p')}(x')\, |\alpha\rangle \otimes |\alpha'\rangle \tag{10.1}$$

together with the antisymmetrized form

$$\begin{aligned} &G_A^{(q,q',p,p')}(x, x', \alpha, \alpha') \\ &\quad = 2^{-1/2}[G_0^{(q,q',p,p')}(x, x', \alpha, \alpha') - G_0^{(q',q,p',p)}(x', x, \alpha', \alpha)] \end{aligned} \tag{10.2}$$

Then one can define an antisymmetrized projector $\Phi_A(C \times C)$ acting upon an antisymmetrized vector $u_A$, replacing Eq. (7.4) by

$$\Phi_A u_A = (2\pi)^{-2} \int_{C \times C} dq\, dq'\, dp\, dp'\, G_A^{(q,q',p,p')}\, |u_A\rangle \tag{10.3}$$

the scalar product involving a summation over spin indices. The generalization is in principle straightforward, whatever the number of electrons and the dimension of phase space. When considering two different cells $C$ and $C'$, it is enough to extend the integral (10.3) to the domain $C \times C'$.

### 10.2. Measurement on One Electron among Two

Let us also go back to measurement theory. Assume, for instance, that we measure a one-electron observable on a two-electron system. The initial state operator will be taken to be $\rho = |\psi\rangle\langle\psi|$, with

$$|\psi\rangle = 2^{-1/2}(|\psi_1\rangle \otimes |\phi_2\rangle - |\phi_1\rangle \otimes |\psi_2\rangle) \tag{10.4}$$

Let us also assume that the ket $|\phi\rangle$ is such that it has a negligible interaction with the measuring apparatus (for instance, it describes a particle far away). Then a trivial reconsideration of the calculations made in Section 9 shows that predicates later than measurement of the observable $A$ at time $t$ are described by representations of logic where the state operator at time $t$ is given by $\rho(t) = |\chi\rangle\langle\chi|$ with

$$|\chi\rangle = 2^{-1/2}[|a_{n1}(t)\rangle \otimes |\phi_2(t)\rangle - |\phi_1(t)\rangle \otimes |a_{n2}(t)\rangle] \tag{10.5}$$

## 11. CONCLUSION

New rules for quantum mechanics have been proposed. They avoid the most questionable assumptions of the conventional approach, such as postulating the possible physical character of every ket or the possibility of measuring every observable. It has been found that, despite the intrinsic probabilistic character of quantum mechanics, classical facts may exist in its framework so that one can consistently discuss individual measurements and give a consistent physical meaning to probabilities.

Some rules of conventional quantum mechanics, such as the rule stating that only the eigenvalues of an observable can occur as a result of an individual measurement, or the rule of wave packet reduction, appear here as the results of a unique and general interpretative rule (Rule 5) having a strictly logical character.

An important logical interdiction rule of conventional theory has been removed: one does not need to assert that, as far as measurement is concerned, a measuring apparatus must be thought of only in terms of classical physics. On the contrary, the logical aspects of classical physics appear as consequences of the quantum interpretative Rule 5.

Quantum mechanics remains of course nonseparable. However, it may be expected that quite a few unwanted epistemological or even broader philosophical consequences of the Copenhagen interpretation[7,10,24,28-31] or of the so-called "orthodox" version[1,32] will not survive in the present approach or that they will be at least appreciably modified.

## APPENDIX. CONSISTENCY CONDITIONS

I first write explicitly all the consistency conditions for a Griffiths family. I shall take the example of four observables $A_j$ ($j = 1,..., 4$), which is enough to exhibit all the properties of the general case. The spectrum $\sigma_j$ of $A_j$ is divided into several subsets $C_j^r$ which will be called here elementary segments. The set of values for the indices $r$, $s$, $q$, and $u$ will be denoted by $R_0, S_0, Q_0$, and $U_0$, respectively, when $j = 1, 2, 3, 4$. With the set $C_j^r$ is associated a projector $E_j^r(t_j)$, which will be written here simply as $E_j^r$, time remaining implicit. Note the property

$$E_j^r E_j^{r'} = \delta_{rr'} E_j^r \tag{A.1}$$

A direct product of $n$ ($n = 4$ here) elementary segments will be called an elementary block. A segment $C_j^R$ is the union (sum) of several elementary segments $C_j^r$, $r \in R$, $R \subset R_0$. It is associated with the projector

$$E_j^R = \sum_{r \in R} E_j^r \tag{A.2}$$

The direct product of $n$ segments will be called a block. Consider such a block $D_0 = C_1^R \times C_2^S \times C_3^Q \times C_4^U$. A special form of condition $C$ occurs when $C_1^R$ is decomposed into elementary segments. It reads

$$\begin{aligned}&\mathrm{Tr}(E_3^Q E_2^S E_1^R \rho E_1^R E_2^S E_1^Q E_4^U)\\&\quad = \sum_{r \in R} \mathrm{Tr}(E_3^Q E_2^S E_1^r \rho E_1^r E_2^S E_1^Q E_4^U)\end{aligned} \tag{A.3}$$

Using Eq. (A.2) for $E_1^R$, developing the sum in the right-hand side of Eq. (A.3), and using Eq. (A.1), one gets

$$\mathrm{Tr}[E_3^Q E_2^S (E_1^r \rho E_1^{r'}) E_2^S E_3^Q E_4^U] = 0; \qquad r \neq r', \quad r \in R, \quad r' \in R \tag{A.4}$$

Here I have used the definition of a double commutator given in Section 2. If these conditions are satisfied, the additivity of projectors [Eq. (A.2)] shows that Condition C of Section 2 is satisfied for any decomposition of $C_1^R$ into a sum of smaller segments, the segments $C_2^S$, $C_3^Q$, $C_4^U$ remaining the same.

Let us now consider Condition C for the decomposition of $C_2^S$ into its constituting elementary segments $C_2^s$, $s \in S$. Here one must not only satisfy the corresponding unicity condition

$$\begin{aligned}\mathrm{Tr}(E_3^Q E_2^S E_1^R \rho E_1^R E_2^S E_1^Q E_4^U) \\ = \sum_{s \in S} \mathrm{Tr}(E_3^Q E_2^s E_1^R \rho E_1^R E_2^s E_3^Q E_4^U)\end{aligned} \qquad (\mathrm{A}.5)$$

but also the consistency conditions (A.4) already established. Since this analysis must hold for any block $D_0$, taking $R = \{r\}$ in Eq. (A.5) gives as before

$$\mathrm{Tr}[E_3^Q (E_2^s E_1^r \rho E_1^r E_2^{s'}) E_3^Q E_4^U] = 0; \qquad s \neq s', \quad s \in S, \quad s' \in S, \quad r \in R_0 \qquad (\mathrm{A}.6)$$

Using Eq. (A.4) in the special case where $S$ is a pair $(s, s')$, $s \neq s'$, developing $E_2^S$ according to Eq. (A.2) and taking Eq. (A.6) into account, one finds that one must have

$$\mathrm{Tr}\{E_3^Q [E_2^s (E_1^r \rho E_1^{r'}) E_2^{s'}] E_3^Q E_4^U\} = 0$$
$$s \neq s', \qquad r \neq r', \qquad s \in S_0, \qquad s' \in S_0, \qquad r \in R_0, \qquad r' \in R_0 \qquad (\mathrm{A}.7)$$

Conversely, the conditions (A.4) and (A.7) entail that any decomposition of $C_1^R \times C_2^S$ into a sum of smaller blocks will satisfy Condition C.

The process is clearly the same when one considers decompositions of $E_3^Q$. On the other hand, a decomposition of $E_4^U$ will give no new condition because the unicity of measure and the validity of the summed consistency conditions will be trivial, exactly as in the simple example given in Section 2.

Finally, the complete set of consistency conditions is given by

$$\begin{aligned}
\mathrm{Tr}\{E_3^q E_2^s [E_1^r \rho E_1^{r'}] E_2^s E_3^q E_4^u\} &= 0 \\
\mathrm{Tr}\{E_3^q [E_2^s E_1^r \rho E_1^r E_2^{s'}] E_3^q E_4^u\} &= 0 \\
\mathrm{Tr}\{[E_3^q E_2^s E_1^r \rho E_1^r E_2^s E_3^{q'}] E_4^u\} &= 0 \\
\mathrm{Tr}\{E_3^q [E_2^s [E_1^r \rho E_1^{r'}] E_2^{s'}] E_3^q E_4^u\} &= 0 \qquad (\mathrm{A}.8) \\
\mathrm{Tr}\{[E_3^q [E_2^s E_1^r \rho E_1^r E_2^{s'}] E_3^{q'}] E_4^u\} &= 0 \\
\mathrm{Tr}\{[E_3^q E_2^s [E_1^r \rho E_1^{r'}] E_2^s E_3^{q'}] E_4^u\} &= 0 \\
\mathrm{Tr}(\{E_3^q [E_2^s [E_1^r \rho E_1^{r'}] E_2^{s'}] E_3^{q'}\} E_4^u) &= 0
\end{aligned}$$

the indices ranging over all $R_0$, $S_0$, $Q_0$, $U_0$, any pair of indices such as $r$ and $r'$ satisfying $r \neq r'$. By construction, these consistency conditions are necessary and sufficient for the measure to be uniquely defined and to satisfy the axioms of probability calculus.

The form initially given by Griffiths for these conditions was

$$\operatorname{Re}\operatorname{Tr}(E_3^Q E_2^S E_1^r \rho E_1^{r'} E_2^S E_3^Q E_4^U) = 0, \qquad r \neq r' \tag{A.9}$$

where $S$, $Q$, and $U$ are any subsets of $S_0$, $Q_0$, and $U_0$, to gether with the analogous conditions one obtains by a permutation, the pair of elementary indices occurring in $R$, $S$, or $U$. This is equivalent to

$$\operatorname{Tr}\{E_3^Q E_2^S [E_1^r \rho E_1^{r'}]\, E_2^S E_3^Q E_4^U\} = 0, \qquad r \neq r' \tag{A.10}$$

as noticed by Griffiths (private communication). Clearly, there are many fewer conditions in (A.8) than in (A.9) and its analogs.

It is much more difficult to discuss the consistency conditions for a general family of propositions not belonging to Griffiths' type (type II). It involves rather tedious considerations in graph theory which would be out of place here. Fortunately, there is apparently in practice only one such kind of family that has useful applications, which I shall call a *special family* and define once again in the case of four observables. Assume that one wants to discuss the elementary predicate associated with a set $D_1 = C_1 \times C_2 \times C_3 \times C_4$ in $X = \sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4$. One can define a convenient basis $\{D_1, D_2, D_3, D_4, D_5\}$ on $X$ as given by

$$\begin{aligned} D_2 &= C_1^* \times C_2 \times C_3 \times C_4, \qquad & D_3 &= \sigma_1 \times C_2^* \times C_3 \times C_4 \\ D_4 &= \sigma_1 \times \sigma_2 \times C_3^* \times C_4, \qquad & D_5 &= \sigma_1 \times \sigma_2 \times \sigma_3 \times C_4^* \end{aligned} \tag{A.11}$$

The history predicates that can be written as a disjunction (union) of elementary predicates are then seen to be given by the sets $D_1 \cup D_2$, $D_1 \cup D_2 \cup D_3$, and so on. The corresponding consistency conditions are then written explicitly as above to become simply

$$\begin{aligned} \operatorname{Tr}\{E_3 E_2 [E_1 \rho E_1^*]\, E_2 E_3 E_4\} &= 0 \\ \operatorname{Tr}\{E_3 [E_2 \rho E_2^*]\, E_3 E_4\} &= 0 \\ \operatorname{Tr}\{[E_3 \rho E_3^*]\, E_4\} &= 0 \end{aligned} \tag{A.12}$$

One should notice how economical this choice is, since one has only $n-1$ consistency conditions for $n$ observables.

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The results of the present work have been written in the form that is most often used in physics, allowing for clearer communication and offering a better aim for criticisms. They have been obtained, however, by using a more systematic approach in the line of modern axiomatization going from logic to mathematics and theoretical physics as recommended by Hilbert. I therefore take this opportunity to thank one of my first masters, Henri Cartan, who gave me the means to follow such a line.

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## REFERENCES

1. J. Von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955).
2. R. Griffiths, *Stat. Phys.* **36**:219 (1984).
3. R. Griffiths, *Am. J. Phys.* **55**:11 (1987).
4. R. Omnès, *C. R. Acad. Sci.* **304**:1039 (1987).
5. R. Omnès, *Phys. Lett. A* **125**:169 (1987).
6. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon Press, Oxford, 1956).
7. B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics*, 2nd ed. (W. A. Benjamin, Reading, Massachusetts, 1976).
8. H. Primas, *Chemistry, Quantum Mechanics and Reductionism* (Springer-Verlag, Berlin, 1981).
9. M. Reed and B. Simon, *Functional Analysis*, Vol. I (Academic Press, New York, 1972).
10. A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**:777 (1935).
11. W. V. Quine, *Mathematical Logic* (Harvard University Press, Cambridge, Massachusetts, 1965).
12. R. P. Feynman, *Rev. Mod. Phys.* **20**:367 (1948).
13. V. Aharonov, P. G. Bergman, and J. L. Lebowitz, *Phys. Rev.* **134B**:1410 (1964).
14. A. Renyi, *Foundations of Probability* (Holden-Day, San Francisco, 1970).
15. B. d'Espagnat, *Phys. Lett. A* **124**:204 (1987).
16. M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974); *The Conceptual Development of Quantum Mechanics* (McGraw-Hill, New York, 1966).
17. H. Weyl, *Bull. Am. Math. Soc.* **56**:115 (1950).
18. C. L. Fefferman, *Bull. Am. Math. Soc* **9**:129 (1983).
19. L. Hörmander, *The Analysis of Linear Partial Differential Operators* (Springer-Verlag, Berlin, 1985).
20. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Ann. Phys.* (N.Y.) **111**:61, 111 (1978).
21. I. Daubechies, unpublished.
22. I. Daubechies, A. Grossmann, and Y. Meyer, *J. Math. Phys.* **27**:1271 (1986).
23. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, Berlin, 1982).

24. E. P. Wigner, *Am. J. Phys.* **31**:6 (1963).
25. W. Heisenberg, *Z. Phys.* **43**:172 (1927).
26. W. Pauli, *Die Allgemeinen Prinzipien der Wellenmechanik*, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1933).
27. L. Van Hove, private communication.
28. H. Margenau, in *Quantum Theory of Atoms, Molecules and the Solid State. A Tribute to John G. Slater*, P. O. Löwdin, ed. (Academic Press, New York, 1966).
29. N. Bohr, *Atomic Physics and Human Knowledge* (Wiley, New York, 1963).
30. É. Schrödinger, *Naturwissenschaften* **23**:807, 823, 844 (1935).
31. D. Bohm, *Quantum Theory* (Constable, London, 1954).
32. F. London and E. Bauer, *La théorie de l'observation en mécanique quantique* (Hermann, Paris, 1939).